Pergamon

PII: S0021-8928(98)00090-2

J. Appl. Maths Mechs, Vol. 62, No. 5, pp. 705–710, 1998

0021-8928/98/$—see front matter

THE MODEL OF DRY FRICTION IN THE PROBLEM OF THE ROLLING OF RIGID BODIES†

V. G. ZHURAVLEV

Moscow

(*Received* 17 *February* 1998)

The Contensou model of combined dry friction [1] is considered. The problem of integrating the shear stresses over the contact area is solved in terms of elementary functions, unlike the solution in [1], reduced to elliptic quadratures. The problem of the rolling of a homogeneous sphere over a plane with dry friction is investigated. © 1999 Elsevier Science Ltd. All rights reserved.

The problem of the rolling of an absolutely rigid body over a surface, with different assumptions regarding the nature of the interaction between them at the contact point, has been solved by many researchers. A detailed description of the present state of the problem and an extensive bibliography can be found in [2].

An analysis of the publications on the subject leads to the conclusion that the solution of this problem, when the interaction at the contact point is described by dry-friction forces, is unsatisfactory. Here we must distinguish between two types of formulations of the problem: (1) it is assumed that the dynamic reactions in the tangential plane at the contact point do not exceed the starting-friction forces, which means that no slippage occurs at this point and hence leads to a non-holonomic relation; (2) slippage is allowed, but its relation to the corresponding reactions is taken in a simplified form. The unsatisfactory nature of both formulations arises from the fact that there are insufficient references to the well-known dry-friction hypothesis (Coulomb's hypothesis) for writing down the conditions at the contact. Coulomb's hypothesis for a point contact have not been formulated, and the conditions which are usually written for it are in fact new hypotheses which do not depend on Coulomb's hypothesis.

Contensou [1] attempted to consider this problem accurately, starting from the natural assumption that, for actual bodies, there is no point contact. According to Hertz's theory of contact stresses, bodies take an elliptic form over the area, at different points of which the slippage is also different. Using Coulomb's hypothesis for an elementary area in side the contact region and integrating over the whole region, Contensou also derived the conditions which a rolling body must satisfy by virtue of this hypothesis. Unfortunately, these conditions are expressed in terms of non-elementary functions, which obviously also explains the lack of problems which have been solved with the same conditions (with the exception of the problem of the Fleuriais gyroscope, which was solved by Contensou himself). Nevertheless, these conditions contain important features, as a result of which the formulations of problems in a simplified form are untenable.

We show below that Contensou's theory allows of a considerable simplification, and its results can be represented in a form which is convenient for solving specific problems.

1. THE MODEL OF DRY FRICTION BETWEEN BODIES IN CONTACT

As in [1], we will assume that the contact between the body and the surface obeys Hertz's contact-stresses theory, described, for example, in [3], and that both contact surfaces are locally spherical. In this case contact occurs over a small circular area of radius ε. This radius depends on the modulus of elasticity of the materials, the applied load N and the radii of curvature of the surfaces at the contact point. The distribution of the normal contact stresses is given by the formula

$$\sigma = \frac{3N}{2\pi\varepsilon^2}\sqrt{1-\frac{\rho^2}{\varepsilon^2}}$$

where ρ is the distance from the centre of the contact circle to the point at which the normal stress σ is defined (Fig. 1).

†*Prikl. Mat. Mekh.* Vol. 62, No. 5, pp. 762–767, 1998.

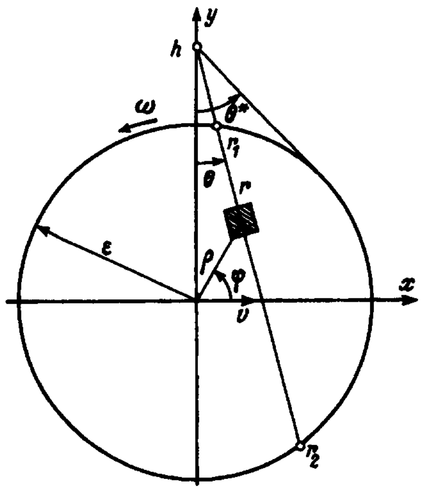

Fig. 1.

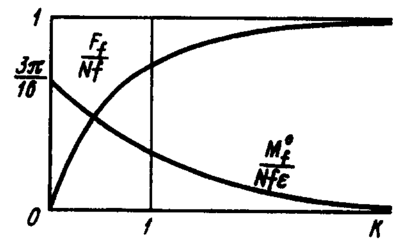

Fig. 2.

We will assume that the relative slippage between the two bodies in the contact area is made up of two simple motions: translational slippage with a velocity υ and pure rotation (pivoting) with angular velocity ω.

The relative velocity at an arbitrary point is perpendicular to the radius vector of this point, drawn from the instantaneous centre of velocities. The elementary friction force is opposite to this velocity and is proportional to the normal stress σ with a coefficient of proportionality equal to the coefficient of dry friction f: $|d\mathbf{F}| = \sigma f\, dx\, dy$.

Integrating the projection of the elementary force dF onto the x axis over the whole area, we obtain the value of the force of sliding friction along the x axis

$$F_f = f\iint \sigma\cos\theta dxdy \tag{1.1}$$

It is clear from symmetry considerations that the integral of the projection of the elementary force onto the y axis is equal to zero, so that (1.1) is the modulus of the principal vector of the tangential friction forces acting at the contact area.

The double integral (1.1) was expressed in [1] in polar coordinates φ, ρ, in which each of the components of the integrals has the form of an elliptic integral. This was the basis for the assertion made in [1, 2] that (1.1) cannot be expressed in terms of elementary functions.

In fact, this is not so. Integral (1.1) can be expressed in terms of elementary functions if we write it in θ, r variables (we further introduce the variable $q = r/\varepsilon$)

$$F_f = \frac{3Nf}{\pi}\int_0^{\theta^*} Q_1(\theta,\ k)d\theta,\quad Q_n(\theta,\ k) = \int_{q_-}^{q_+}\sqrt{-q^2+2qk\cos\theta+1-k^2}\,q^n dq$$
$$\sin\theta^* = \varepsilon/h = \varepsilon\omega/v,\quad q_\pm = k\cos\theta \pm\sqrt{1-k^2\sin^2\theta},\quad k = h/\varepsilon \tag{1.2}$$

The expression for the friction force has the form (1.2) if the instantaneous centre of velocities is outside the contact area, i.e. when $k > 1$. If $k \leqslant 1$, we have instead of (1.2)

$$F_f = \frac{3Nf}{2\pi}\int_0^{\pi} Q_1(\theta, k)d\theta$$

Both of these integrals can be evaluated in terms of elementary functions

$$F_f = Nf\times\begin{cases}\dfrac{3}{32}\pi k(4-k^2), & k\leqslant 1\\ \dfrac{3}{64k}[4k^2(4-k^2)\theta^* + 4(k^2+2)\sqrt{k^2-1}], & k>1\end{cases} \tag{1.3}$$

A graph of the function $F_f(k)$ (1.3) is shown in Fig. 2.

It can be verified that both the first derivative of the function F_f and the function itself are continuous at the point $k = 1$. Note also the behaviour of the function (1.3) for small and large k

$$F_f = \frac{3}{8}\pi Nfk + O(k^2), \quad F_f = Nf\left(1 - \frac{1}{10k^2}\right) + O\left(\frac{1}{k^3}\right)$$

To use the friction law (1.3) in problems of the dynamics of rolling rigid bodies, a fractional-linear approximation of function (1.3) (the Padé approximation) is the most convenient, since this preserves the derivative at zero and the limit at infinity

$$F_f = Nf\frac{3\pi k}{8 + 3\pi k} = Nf\frac{3\pi \upsilon}{8\varepsilon\omega + 3\pi\upsilon} \tag{1.4}$$

Function (1.4), which accurately gives the qualitative relationship between the friction force and the slippage velocity υ and the pivoting velocities ω, approximates function (1.3) quantitatively fairly well also.

Contensou [1] confined himself to calculating the principal vector of the friction forces. However, for a complete representation of the conditions at the contact it is also necessary to calculate the principal moment of these forces about the centre of the contact area: $M_f = M_f^h - hF_f$, where M_f is the principal moment of the forces about the instantaneous centre of velocities

$$M_f^h = \frac{3Nf\varepsilon}{2\pi} \times \begin{cases} \int\limits_0^{2\pi} Q_2(\theta, k)d\theta, & k \leq 1 \\ 2\int\limits_0^{\theta^*} Q_2(q, k)d\theta & k > 1 \end{cases}$$

These integrals can also be easily evaluated

$$M_f^\circ = \frac{3Nf\varepsilon}{128} \times \begin{cases} \pi(8 - 8k^2 + 3k^4), & k \leq 1 \\ 2[(8 - 8k^2 + 3k^4)\theta^* + 3(2 - k^2)\sqrt{k^2 - 1}, & k > 1 \end{cases} \tag{1.5}$$

Function (1.5) is shown in Fig. 2. Like the function (1.3) it is continuous at the point $k = 1$.

The Padé approximation, which preserves the value at zero and the behaviour at infinity of the moment of the pivoting friction, has the form

$$M_f^\circ = \frac{3\pi Nf\varepsilon}{16 + 15\pi k} = \frac{3\pi Nf\varepsilon^2\omega}{16\varepsilon\omega + 15\pi\upsilon} \tag{1.6}$$

The expressions obtained for the sliding friction force (1.3) and (1.4) and the moment of the pivoting friction (1.5) and (1.6), based on the use of Coulomb's dry friction hypothesis, enable us to draw the following conclusions, which are fundamental when using them in problems of the dynamics of rolling bodies.

1. Sliding friction and pivoting friction are not independent of one another. The sliding friction force F_f is a function of both the sliding velocity υ and the angular pivoting velocity ω. The moment of pivoting friction M_f is also a function of these two arguments. Hence it follows that the conditions, often employed, in which these components are independent of one another, are speculative and bear no relation to Coulomb's dry-friction model.

2. The idea of starting friction, characteristic for the one-dimensional Coulomb friction model, does not apply in general. For any non-zero pivoting velocity ω, sliding friction F_f behaves like viscous friction in the neighbourhood of slow sliding velocities. A similar situation also occurs for pivoting friction M_f. For this reason such a well-known example of non-holonomic mechanics as the problem of the rolling of body without slipping at the contact point is based on an incorrect representation of the laws of dry friction in complex motion. In fact, Coulomb dry friction cannot lead to a non-holonomic relation.

3. The functions (1.3)–(1.6) have no limit at the point $\upsilon = \omega = 0$. This means that, without a priori information on the nature of the rolling, any further simplification of models (1.4) and (1.6) is impossible.

This big difference in the manifestation of dry friction, based on Coulomb's hypothesis, from those simplified representations of it which occur in innumerable solved problems on the rolling of rigid bodies, and which do not follow from Coulomb's dry-friction hypothesis, naturally lead to a distorted representation of the behaviour of the bodies in these problems. In those cases when the results of the problems solved with these conditions allow of an experimental check, it turns out that experiment contradicts the theory.

A more careful analysis of the friction conditions undertaken by Contensou [1], was similarly stimulated by the lack of agreement with some of the known models (the absence of slippage, leading to a non-holonomic formulation or the presence of slippage with the one-dimensional model of dry friction) and the multiply repeated experiment with Fleuriais gyroscope.

The problem of the rolling of a homogeneous sphere under dry-friction conditions, defined by (1.4) and (1.6), investigated below, leads to results which differ considerably from existing results for this problem (see, for example, [2]).

2. ROLLING WITH DRY FRICTION OF A HOMOGENEOUS HEAVY SPHERE ALONG A HORIZONTAL PLANE

The equations of motion of a homogeneous sphere of radius R and mass m (Fig. 3)

$$J\dot{\boldsymbol{\omega}} = \mathbf{M}, \quad m\ddot{\mathbf{r}} = \mathbf{F}; \quad J = \tfrac{2}{5}mR^2$$

will be considered in the projections onto fixed axes.

We will express the velocity of a point on the sphere, coinciding with the centre of the contact point

$$v_x = \dot{x} - R\omega_y, \quad v_y = \dot{y} + R\omega_x \tag{2.1}$$

using polar variables

$$v = \sqrt{v_x^2 + v_y^2} \geqslant 0, \quad \cos\alpha = v_x / v, \quad \sin\alpha = v_y / v \tag{2.2}$$

From (1.4) we obtain the following expressions for the components of the force $\mathbf{F}$

$$F_x = -F\cos\alpha, \quad F_y = -F\sin\alpha; \quad F = \frac{3\pi N f v}{8\varepsilon|\omega_z| + 3\pi v} \tag{2.3}$$

The components of the moment of the friction forces acting on the body around its centre are

$$M_x = RF_y, \quad M_y = -RF_x, \quad M_z = \frac{3\pi N f \varepsilon^2 \omega_z}{16\varepsilon|\omega_z| + 15\pi v} \tag{2.4}$$

The last component is written using (1.6).

The complete system of equations of the dynamics of the rolling of a body with dry friction is

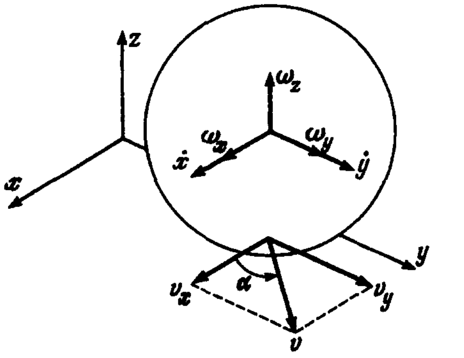

Fig. 3.

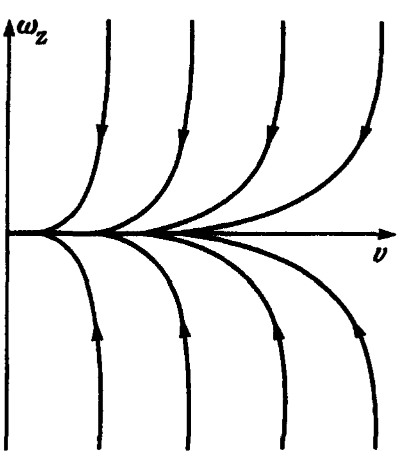

Fig. 4.

$$J\dot{\omega}_x = RF_y, \quad J\dot{\omega}_y = -RF_x, \quad J\dot{\omega}_z = -M_z$$
$$m\ddot{x} = F_x, \quad m\ddot{y} = F_y \tag{2.5}$$

taking (2.1)–(2.4) into account.

System (2.5) depends solely on the velocities (angular and linear) and is independent of the variables which define the position and orientation of the body. It will henceforth be more convenient to change from the variables ω_x, ω_y, ω_z, $\dot{x}$, $\dot{y}$ to the variables ω_x, ω_y, u, υ, α, using (2.1) and (2.2), and also to put $u = R\omega_z$ and $\mu = \varepsilon/R$. If, after this, we make the replacement of the time $t \to \tau = (3\pi Nf/(2m))t$, Eqs (2.5) in the new variables reduce to the form

$$\frac{d\omega_x}{d\tau} = -\frac{5v\sin\alpha}{R(8\mu|u|+3\pi v)}, \quad \frac{d\omega_y}{d\tau} = \frac{5v\cos\alpha}{R(8\mu|u|+3\pi v)}$$
$$\frac{du}{d\tau} = -\frac{5\mu^2 u}{16\mu|u|+15\pi v}, \quad \frac{dv}{d\tau} = -\frac{7v}{8\mu|u|+3\pi v}, \quad \frac{d\alpha}{d\tau} = 0 \tag{2.6}$$

Hence, it quickly follows that the direction of the relative velocity of slippage does not change during the motion of the sphere: $\alpha \equiv \text{const}$. Without loss of generality, we can assume that $\alpha \equiv 0$, which gives $\omega_x = \text{const}$ and $7R\omega_y + 5\upsilon = \text{const}$.

Hence, the solution of the system of five equations has been reduced to the solution of two equations in the variables u and υ. If the latter is solved, all the remaining ones are found in quadratures

$$\omega_y = \frac{1}{7R}(\text{const} - 5v), \quad x = \frac{1}{7}\int(2v + \text{const})dt, \quad y = y_0 - R\omega_x t$$

We will first investigate the behaviour of the system qualitatively. To do this we write the equations of the integral curves in the u, υ plane

$$\frac{dv}{du} = \frac{7v(16\mu|u|+15\pi v)}{5\mu^2 u(8\mu|u|+3\pi v)}$$

The integral curves are shown in Fig. 4.

Henceforth, without loss of generality, it is sufficient to consider the case $u \geqslant 0$. The following system with an elliptic right-hand side has exactly the same integral curves

$$\frac{du}{d\beta} = -5\mu^2 u(8\mu u + 3\pi v), \quad \frac{dv}{d\beta} = -7v(16\mu u + 15\pi v) \tag{2.7}$$

for which all the curves reach equilibrium positions after an infinite time in the variable β. The relation between the independent variables β and τ is given by the equation

$$\tau = \int_0^\beta (16\mu u + 15\pi v)(8\mu u + 3\pi v)d\beta \tag{2.8}$$

System (2.7) has the particular solution $u = u(1)/\beta$, $\upsilon = \upsilon(1)/\beta$. The behaviour of all the other solutions of this system $u \sim 1/\beta$, $\upsilon \sim 1/\beta$ as $\beta \to \infty$ is the same. Hence it follows that integral (2.8) converges as $\beta \to \infty$. This means that all the integral curves of system (2.6) in the (u, υ) plane arrive at the point $u = \upsilon = 0$ after a finite time. Hence, the slippage velocity υ and the pivoting velocity $\omega_z = u/R$ simultaneously vanish. The angular velocity ω_y becomes constant from this instant. Further motion consists of rolling without sliding along a straight line with constant linear and angular velocity, and the angular velocity lies in the (x, y) plane and is perpendicular to the linear velocity.

An analytical solution of system (2.6), which is interesting during the finite time interval during which slippage occurs, can be constructed by changing to canonical coordinates of the similarity group $u \to u' = au$, $\upsilon \to \upsilon' = a\upsilon$ which, as can easily be seen, is the group symmetry of system (2.6). The corresponding replacement $(u, \upsilon) \to (q, p)$, where $q = \upsilon/u$, $p = \ln \upsilon$, leads to an equation with separable variables.

The whole problem has thereby been reduced to quadratures.

The analytical solution can be simplified considerably using the smallness of the dimensionless parameter $\mu = \varepsilon/R$ by finding a solution in the form of appropriate asymptotic forms in μ.

REFERENCES

1. CONTENSOU, P., Couplage entre frottement de glissement et frottement de pivotement dans la théorie de la toupie. *Gyrodynamics*. Symposium Celerina, August 20–23, 1962, Springer, 1963.
2. MARKEYEV, A. P., *The Dynamics of a Body in Contact with a Solid Surface*. Nauka, Moscow, 1992.
3. PONOMAREV, S. D., BIDERMAN, V. L., LIKHAREV, K. K. *et al.*, *Strength Calculations in Machine Construction*. Vol. 2. Mashgiz, Moscow, 1958.

Translated by R.C.G.